Consistency, Stability and Convergence on Nonuniform Grids in Elliptic BVPs

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Motivation

- Singularly perturbed BVPs with boundary layers:
  fine grid resolution at the boundary layer, coarse grid elsewhere
  $\rightsquigarrow$ nonuniform grids
- Is the Shishkin mesh good enough? Maybe not: see Söderlind and Yadaw for the details
- Minimal question:

$$\begin{cases} -u'' = f \\ u(0) = 0,\ u(1) = 0 \end{cases}$$

What are the good grids for this simple problem?

Uniform vs nonuniform grids

$$\begin{cases} -u'' = f \\ u(0) = 0,\ u(1) = 0 \end{cases} \qquad FDM \quad \rightsquigarrow \quad \mathbf{A}_N \mathbf{u}_N = \mathbf{f}_N$$

Uniform grid

$$\frac{-u_{n-1} + 2u_n - u_{n+1}}{h^2} = f_n$$

Nonuniform grid

$$\frac{-2u_{n-1}}{h_- (h_- + h_+)} + \frac{2u_n}{h_- h_+} + \frac{-2u_{n+1}}{h_+ (h_- + h_+)} = f_n$$

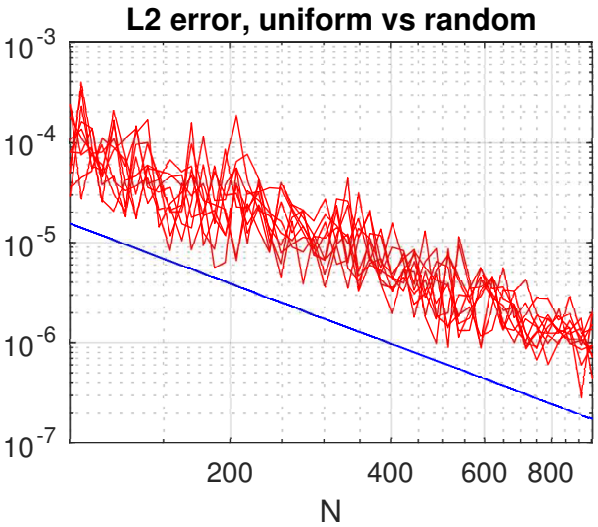
Lax theory

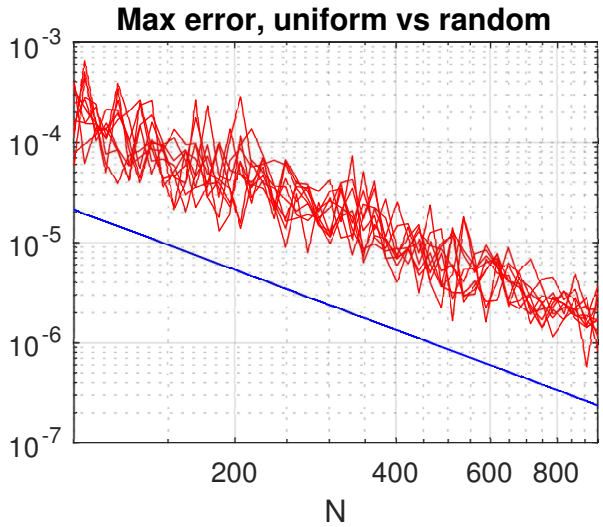
- Consistency + Stability = Convergence
- Order of consistency $\stackrel{?}{=}$ order of convergence
- Uniform grid: easy!
- Nonuniform grid: not so easy!

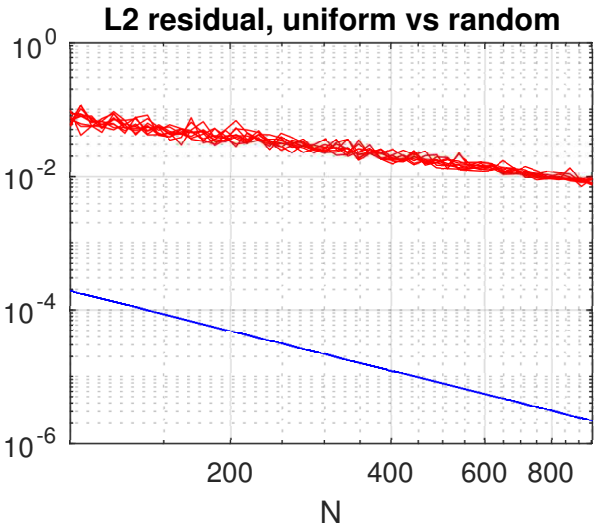
Nonuniform grid: Consistency

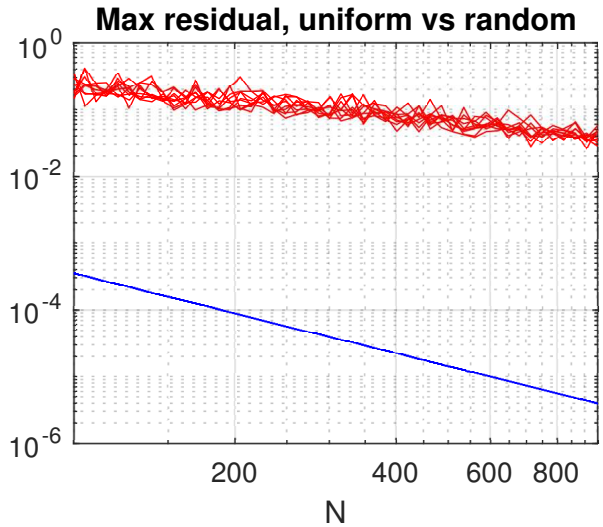
$$|r_n| = \mathrm{O}\left(|h_+ - h_-|\right) \approx \frac{\phi'}{\phi^3} h^2 \neq \mathrm{O}\left(h^2\right)$$

The beginning: random grids









In spite of that the order of consistency is only 1 the order of convergence is 2.

$$\|\mathbf{e}_N\| \leq \left\|\mathbf{A}_N^{-1}\right\| \|\mathbf{r}_N\|$$

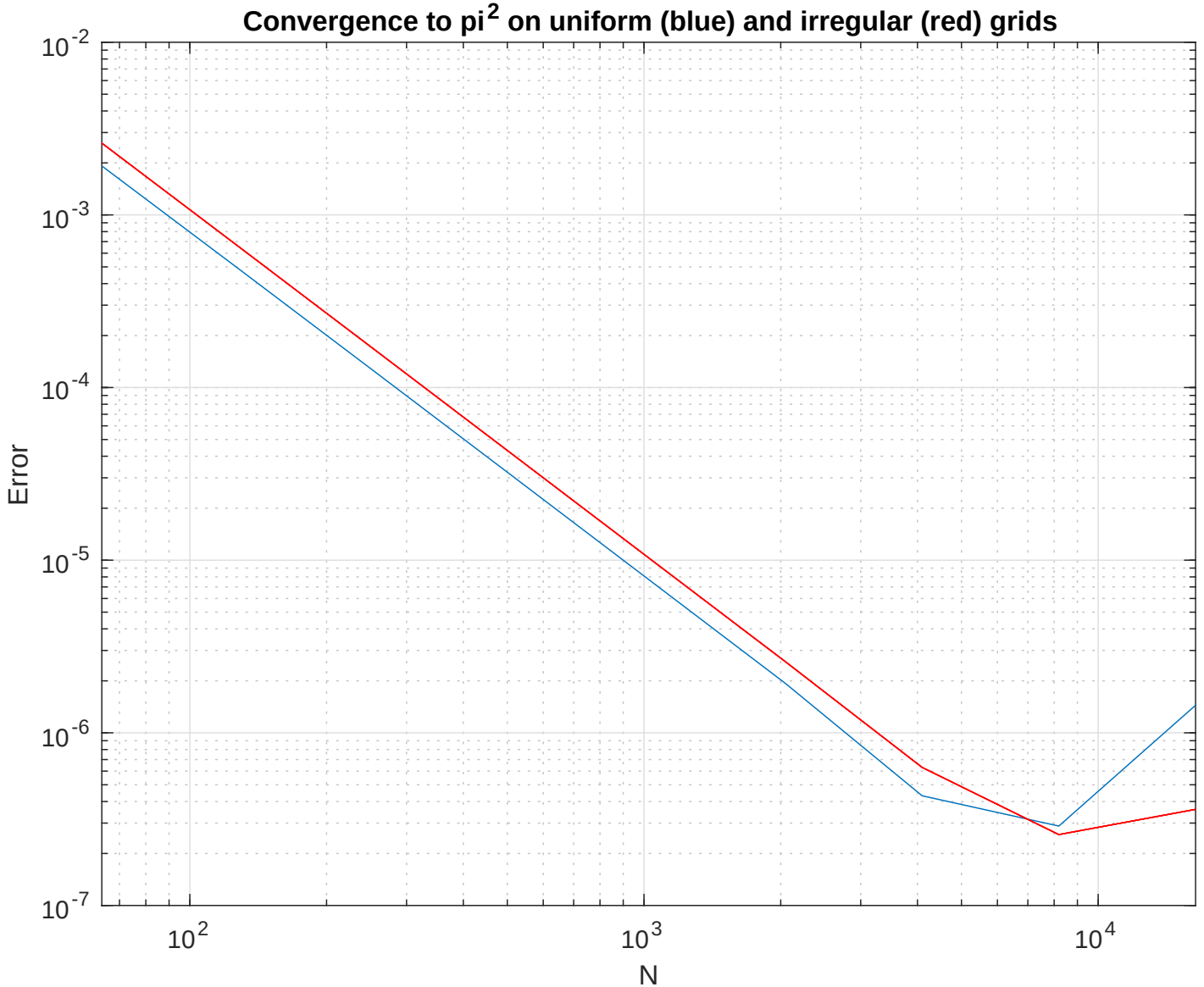
is not enough! Possible reasons: $\mathbf{A}_N^{-1}$ is strongly regularizing.

$$\mathbf{A}_N = \mathbf{V}_N \mathbf{D}_N \mathbf{V}_N^{-1}$$

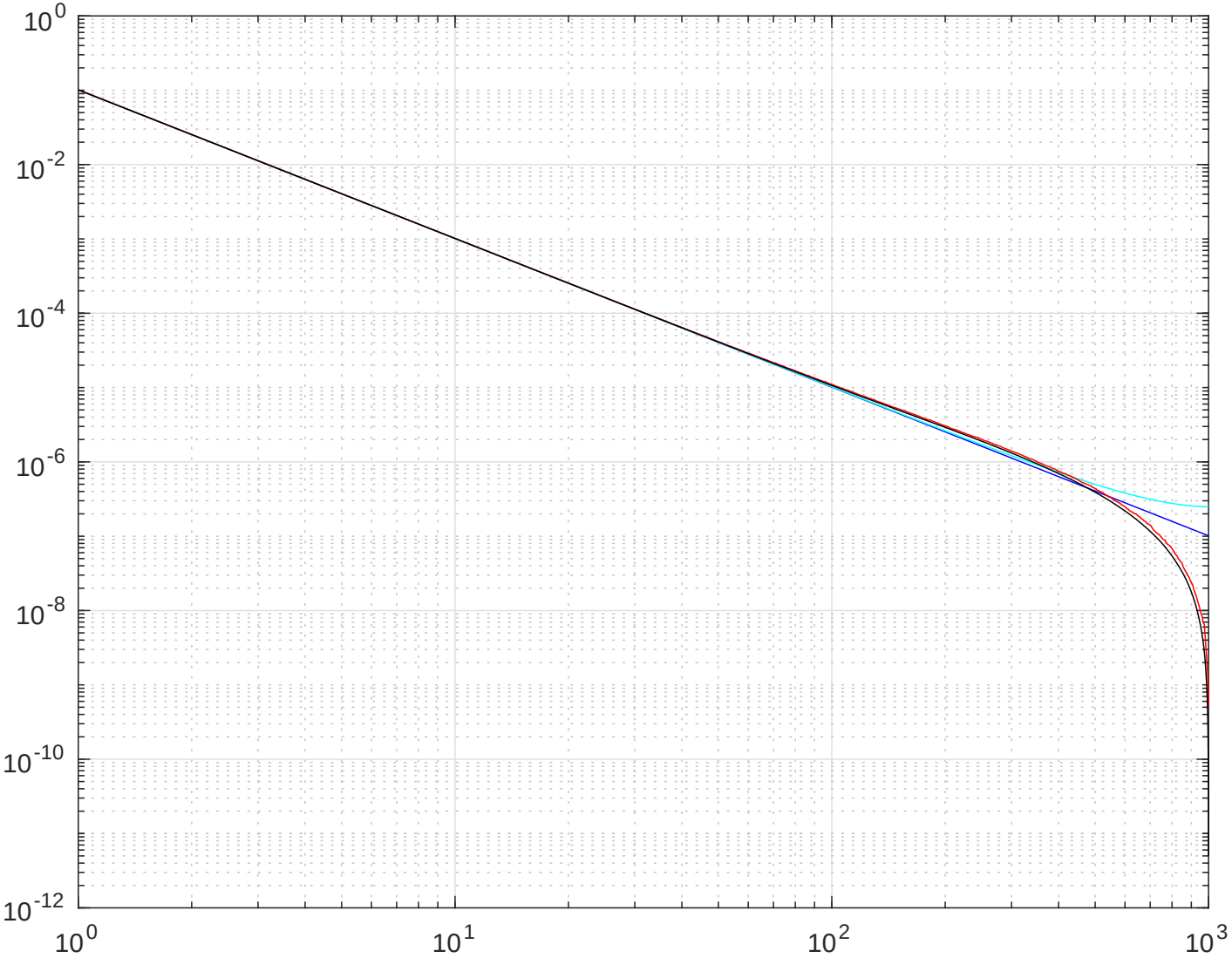
- $\mathbf{V}_N$ normalized eigenvectors
- $\mathbf{D}_N$ eigenvalues

$$\mathbf{r}_N = \sum_{i=1}^{N} c_i \mathbf{v}_i \quad \rightsquigarrow \quad \|\mathbf{e}_N\| = \| \sum_{i=1}^{N} \frac{c_i}{\lambda_i} \mathbf{v}_i \| \leq \sum_{i=1}^{N} \left| \frac{c_i}{\lambda_i} \right| \|\mathbf{v}_i\|$$

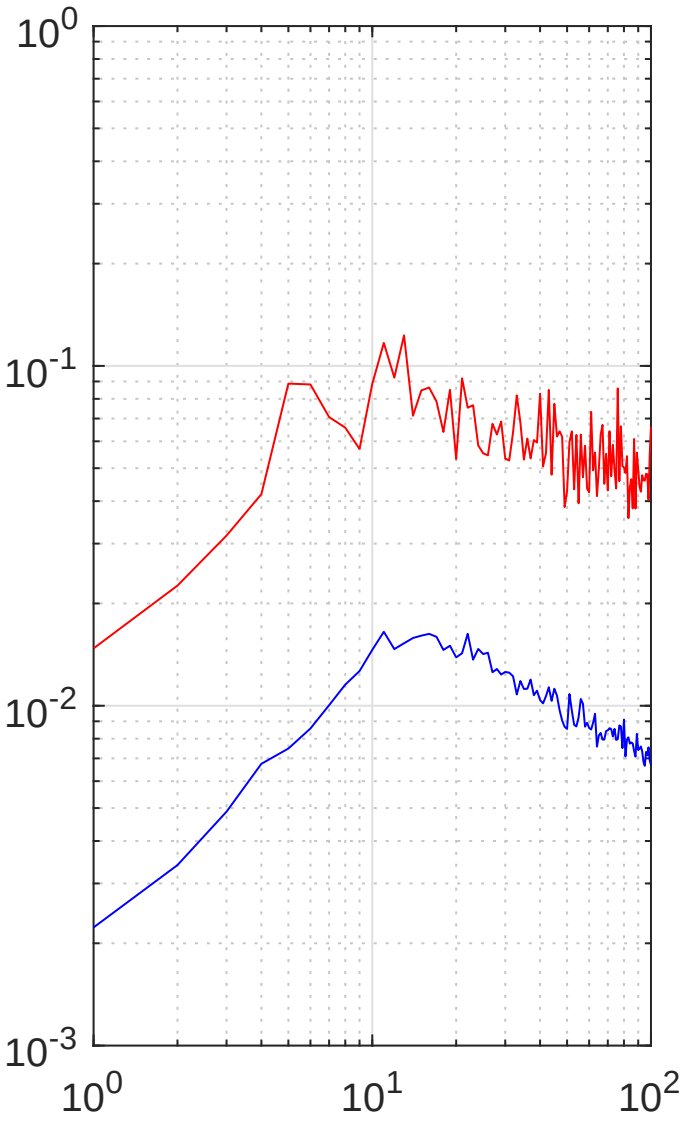
Eigenvalues of $\mathbf{A}_N^{-1}$

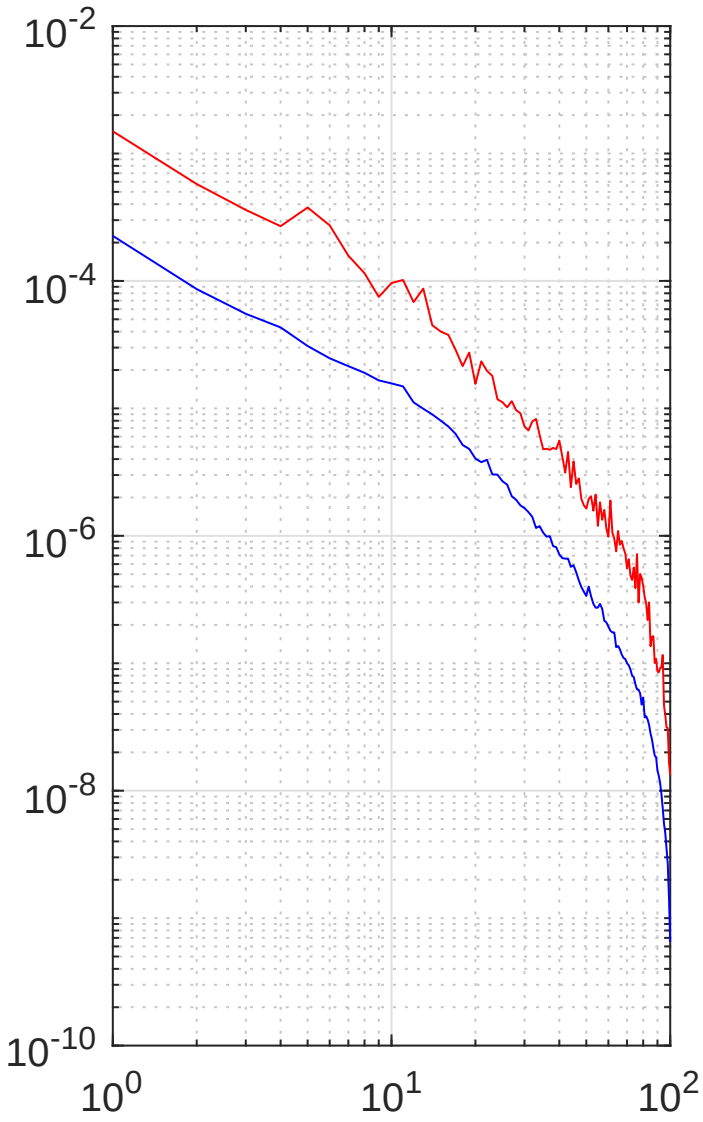


Eigenvalues of $\mathbf{A}_N^{-1}$

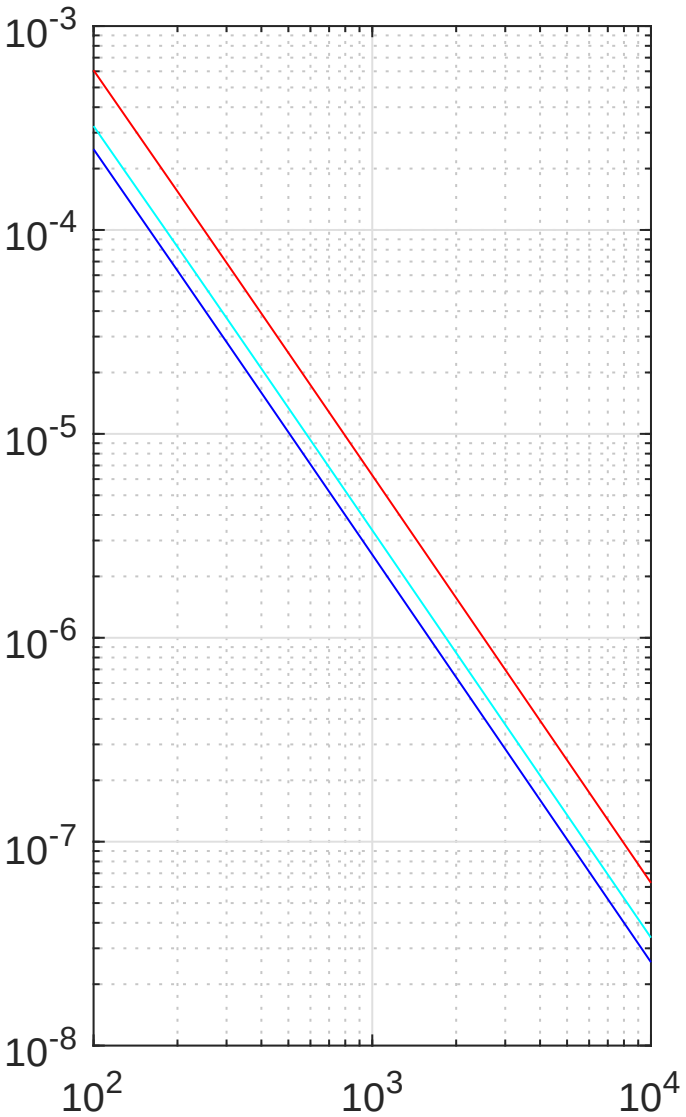


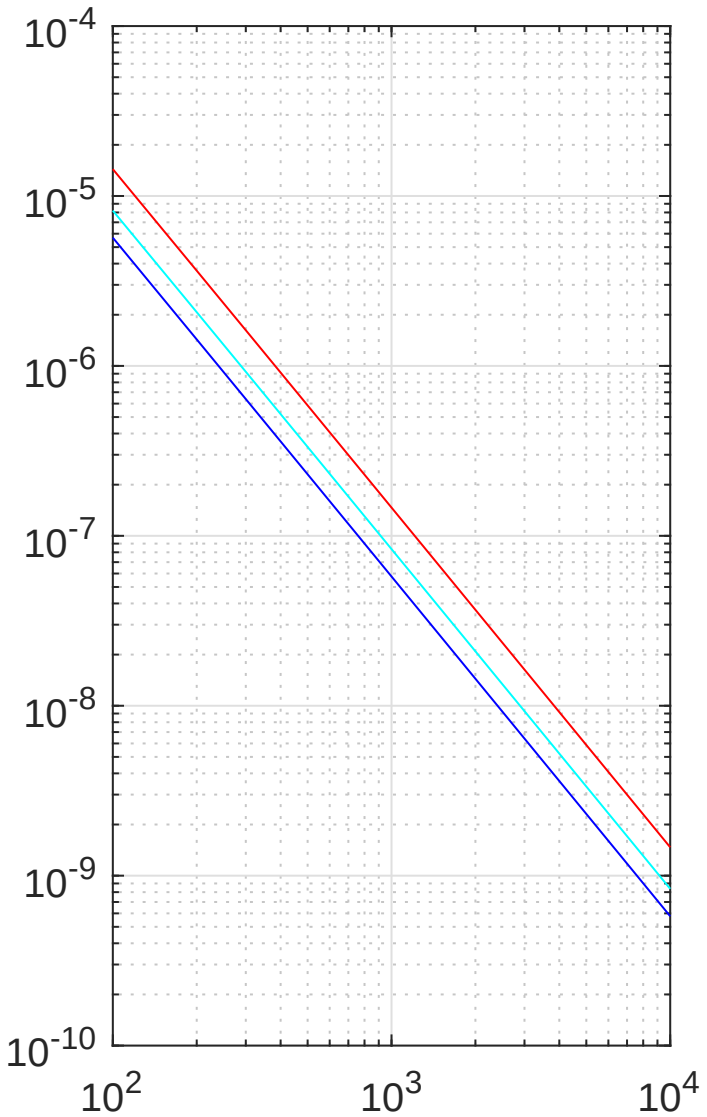
Coefficients





x^α grids: $\alpha = 1/3$





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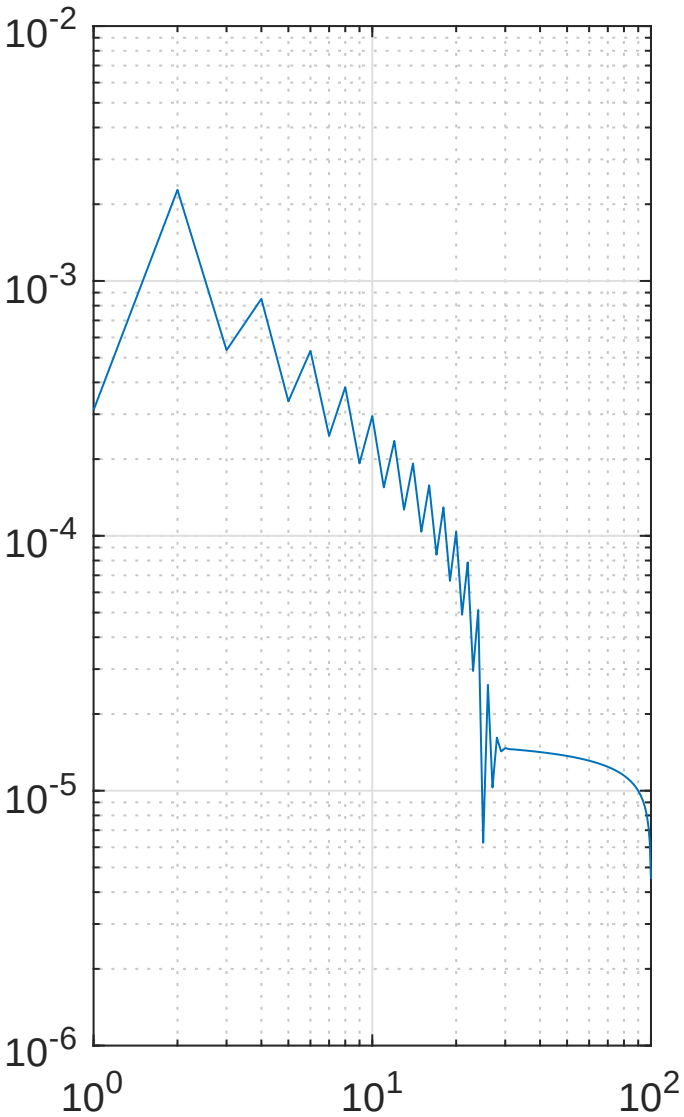
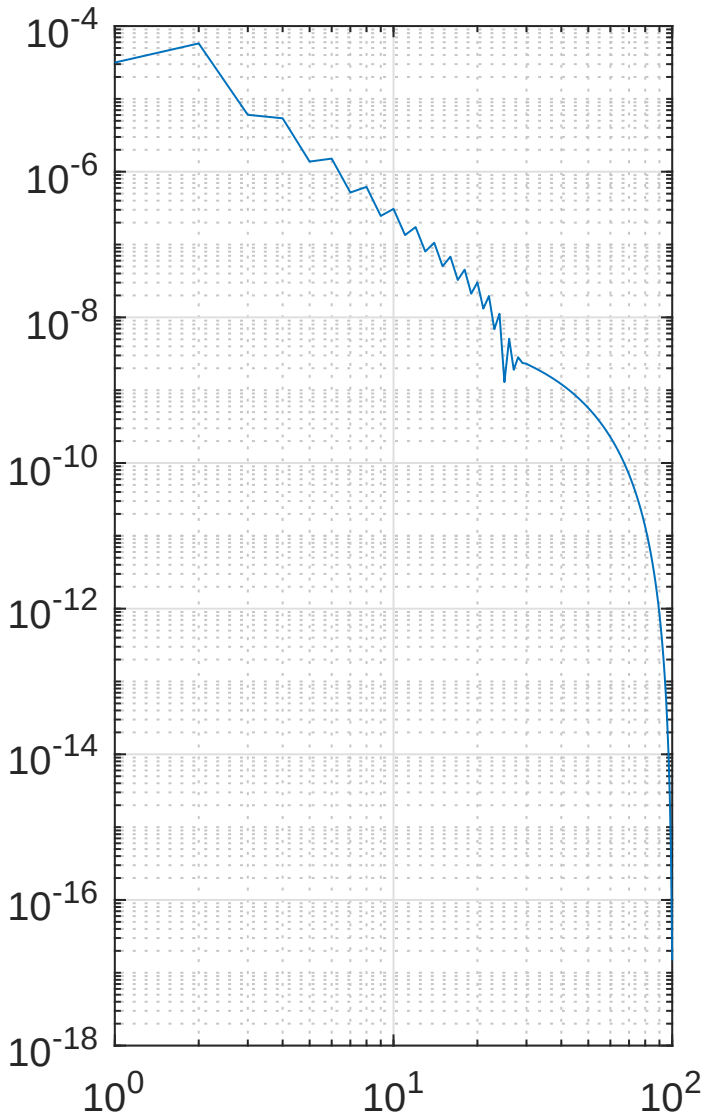
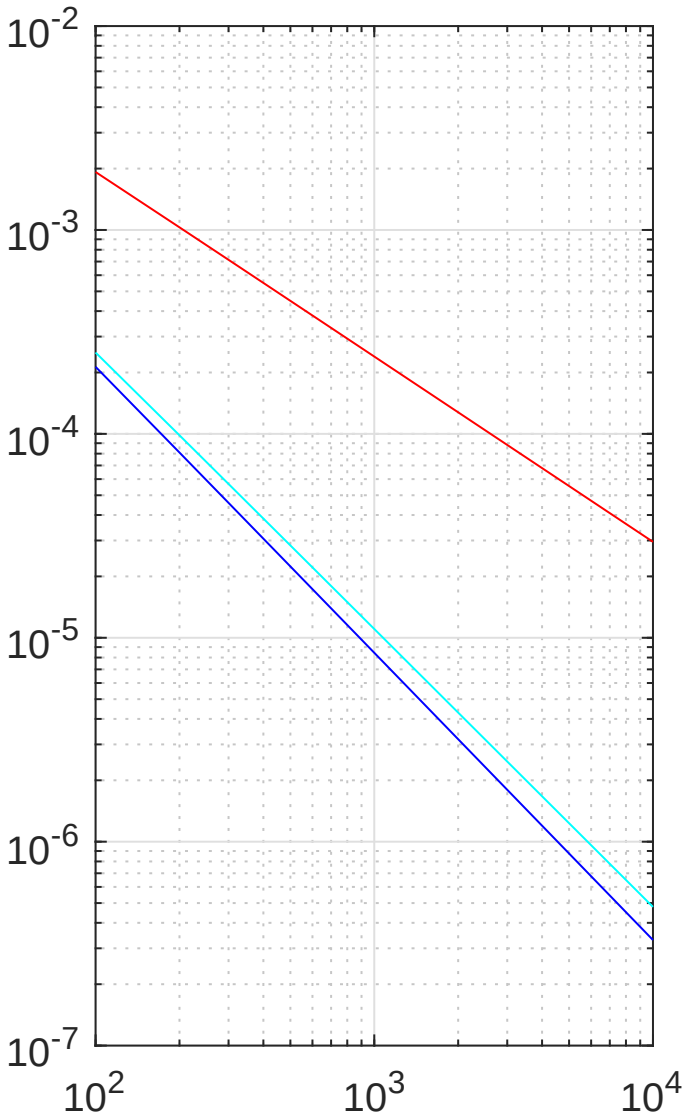
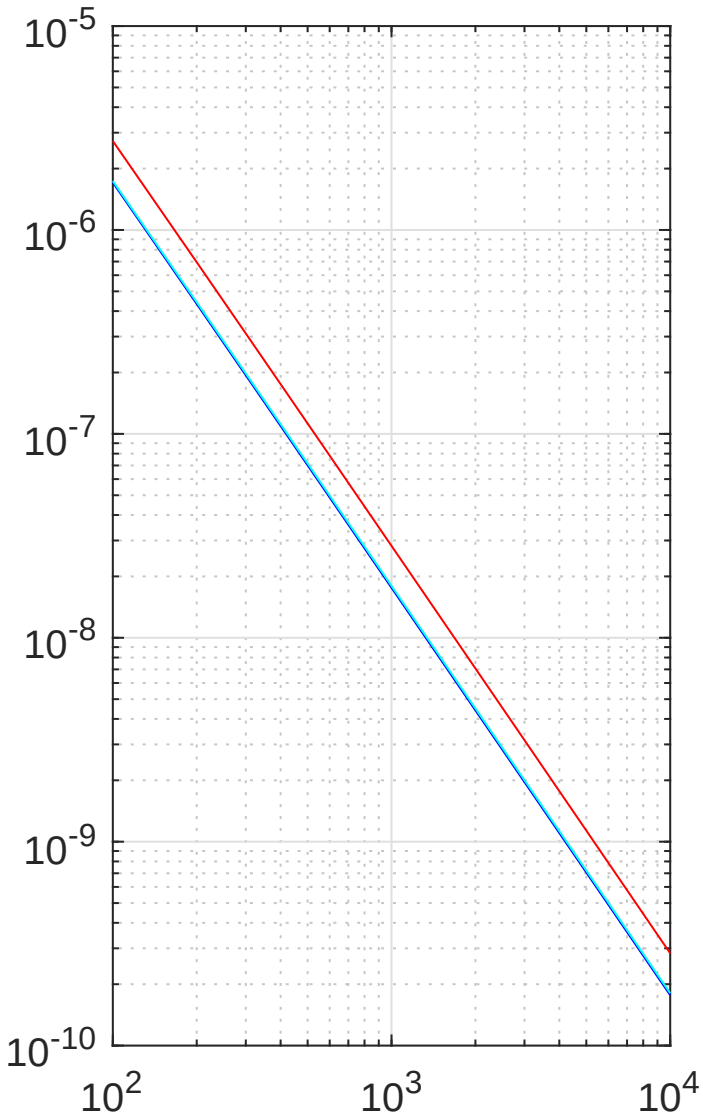




x^α grids: $\alpha = 11/10$





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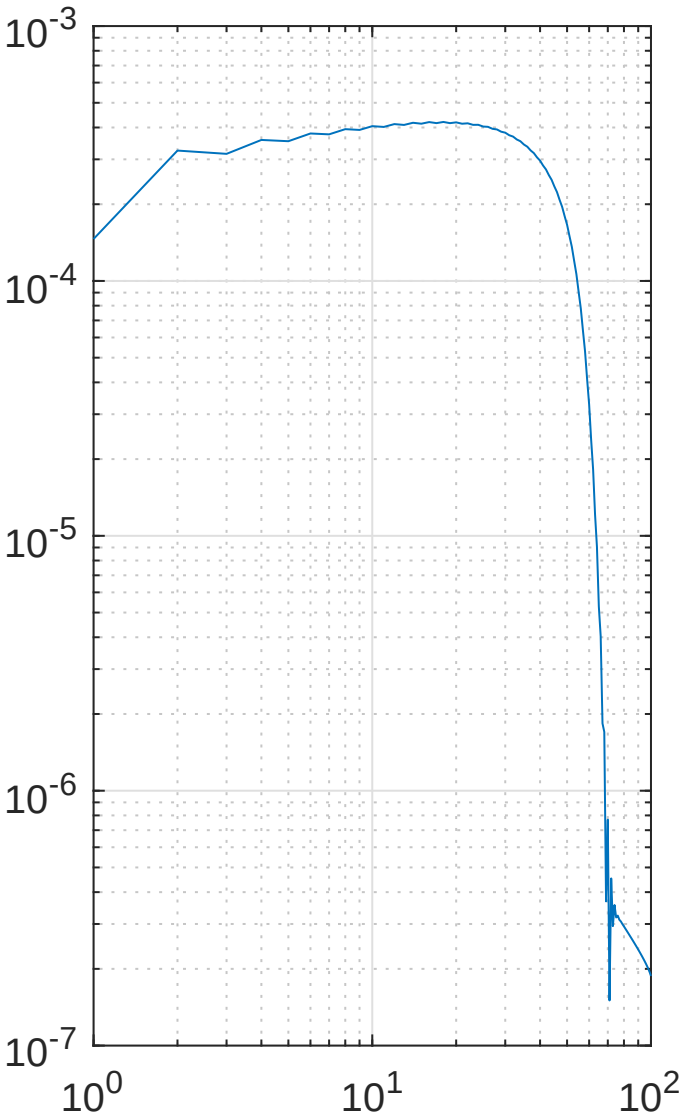
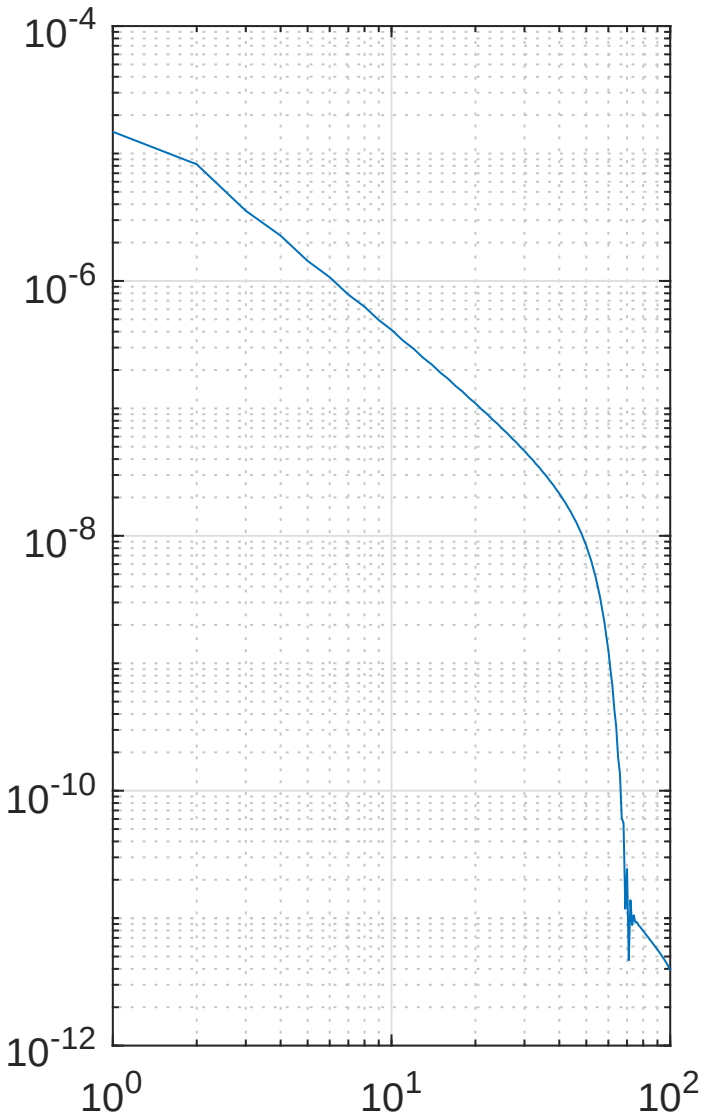
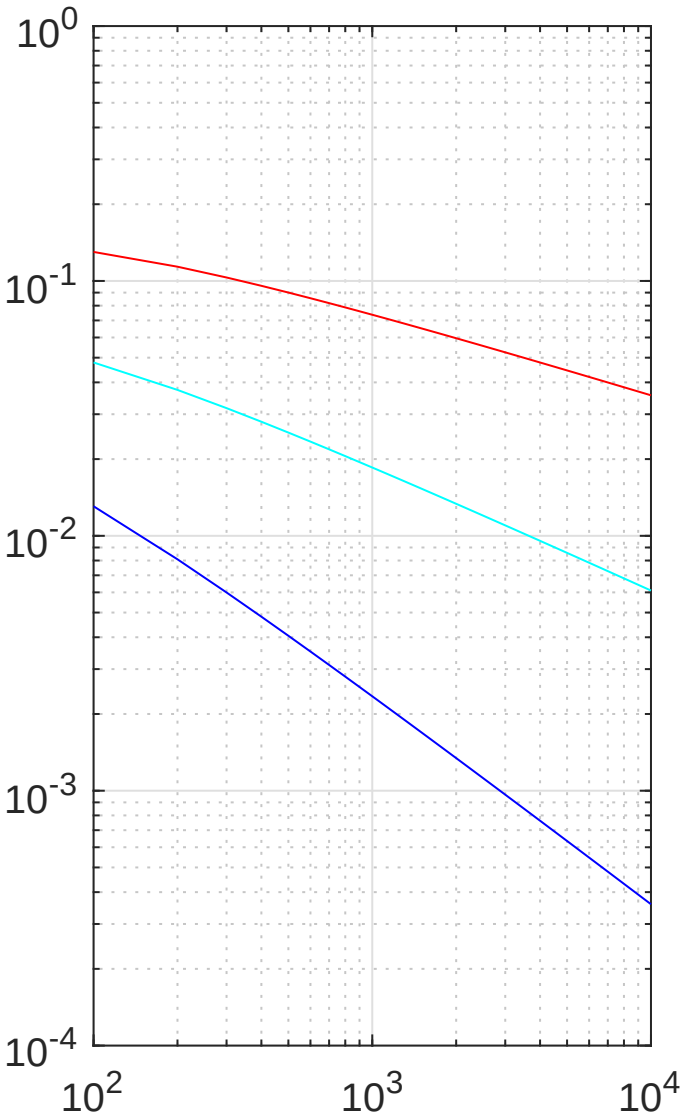
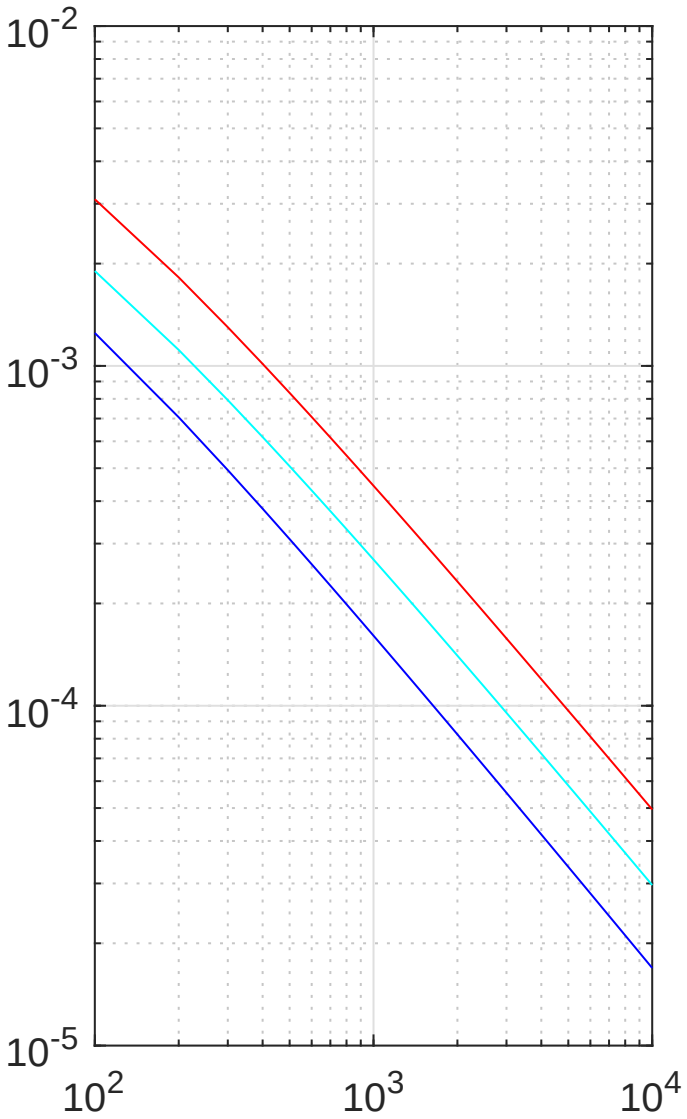




x^α grids: $\alpha = 3$





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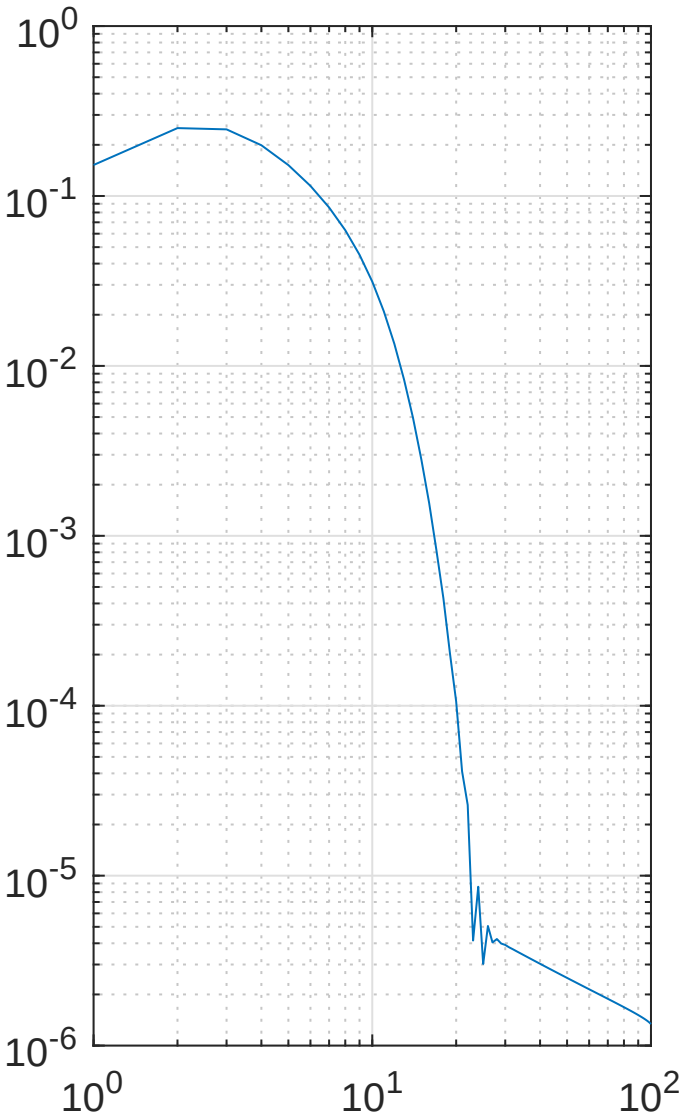
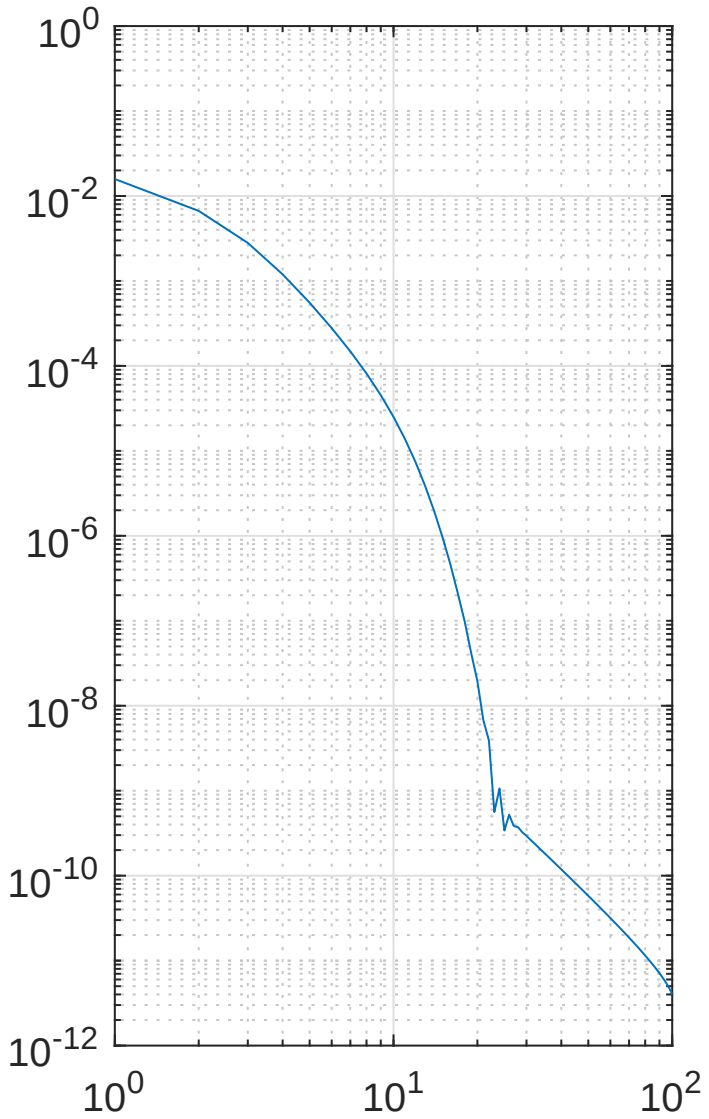
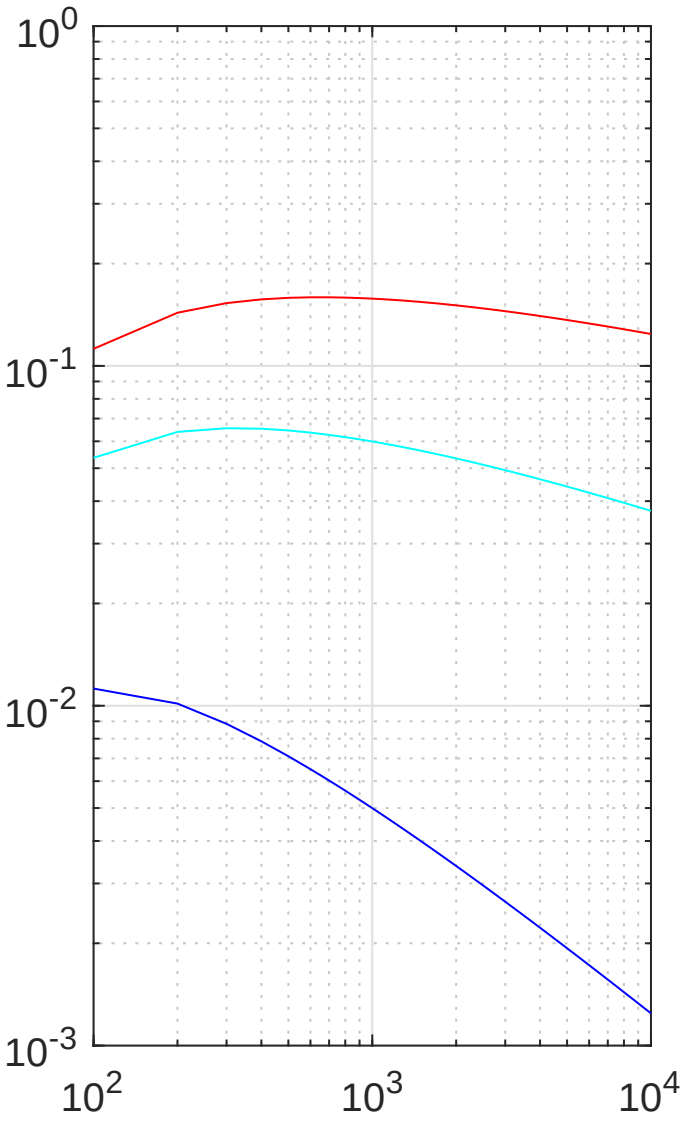
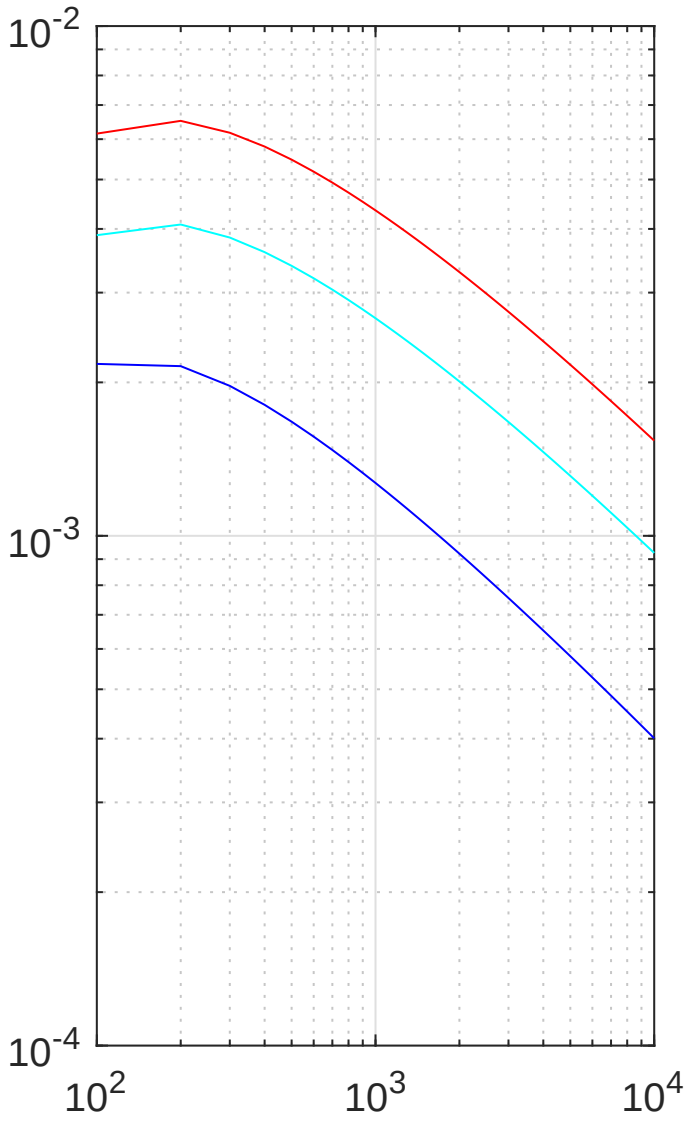




x^α grids: $\alpha = 5$





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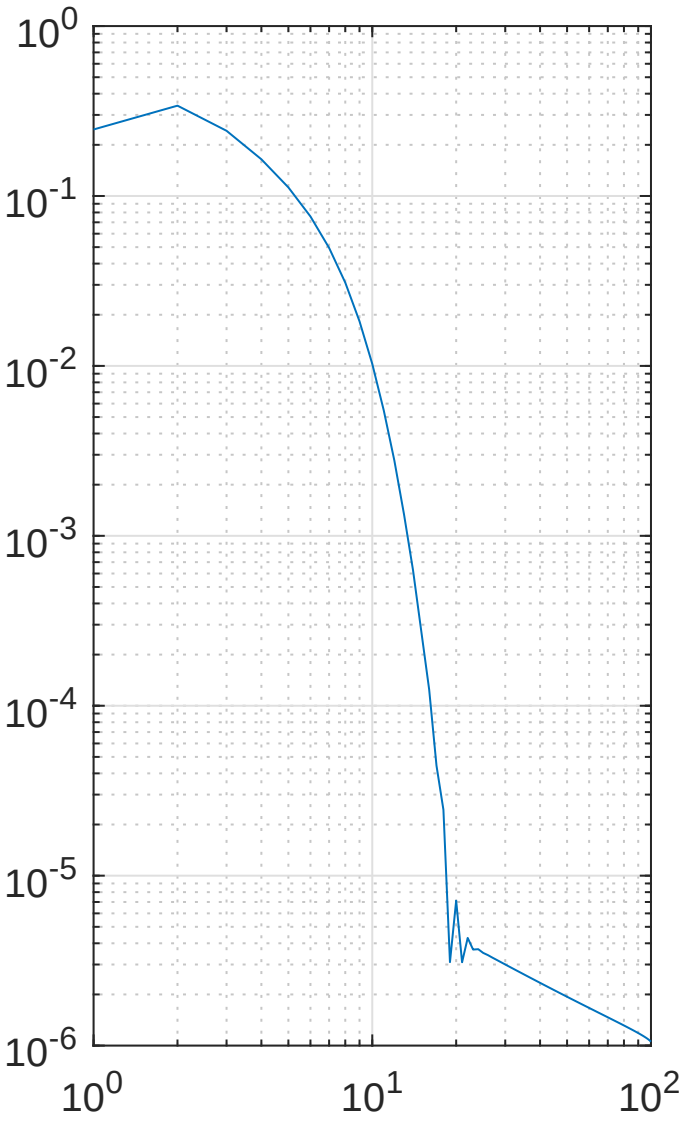
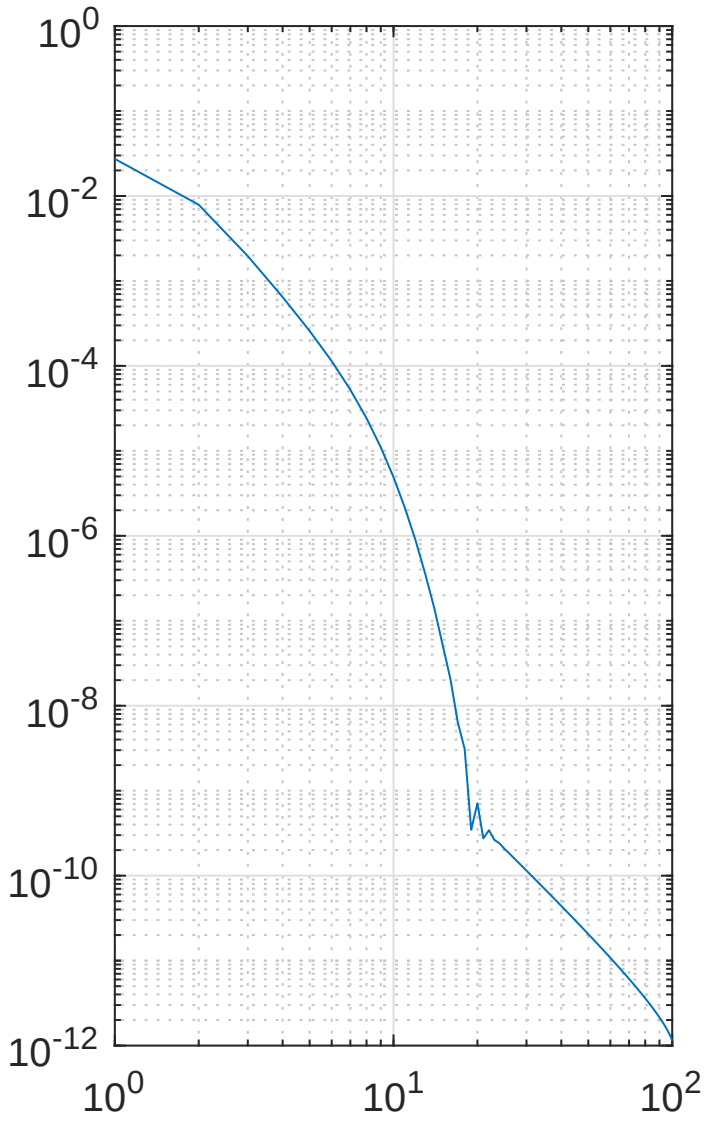
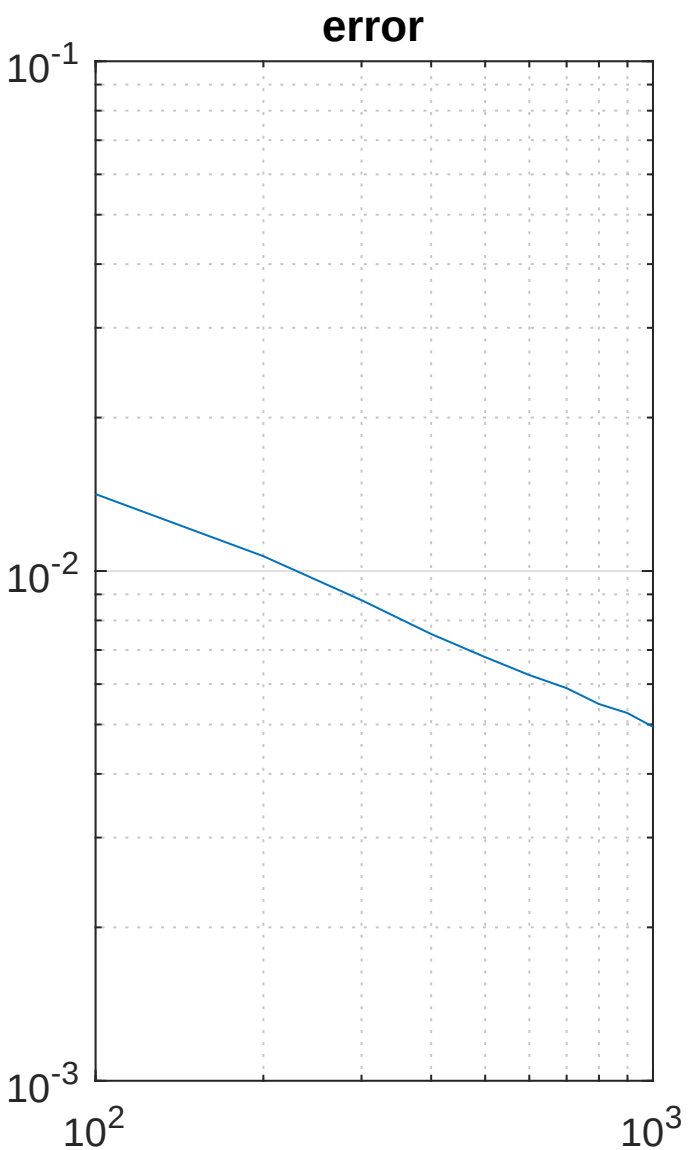
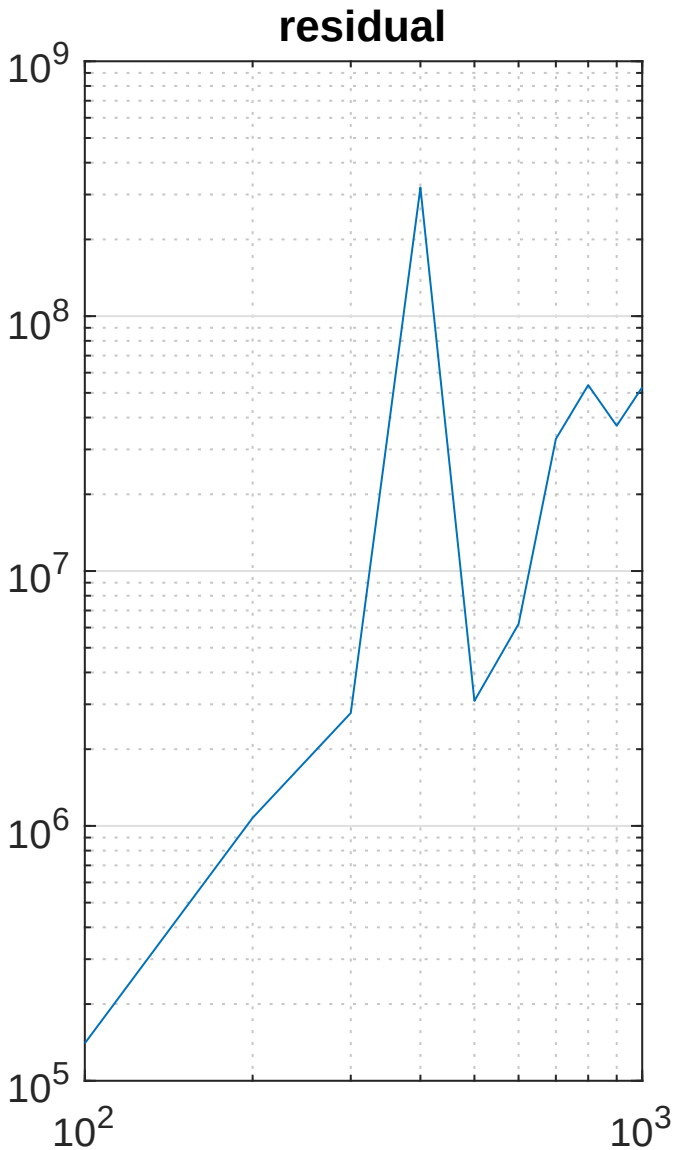




?

$\mathbf{A}_{N,random}\mathbf{u}_N = \mathbf{f}_N$





Thank you for your attention!